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# Elegant Iterative Methods for Solving a Nonlinear Matrix Equation $X - A^*e^XA = I$

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**Abstract**

The nonlinear matrix equation $X - A^*e^XA = I$ was solved by Gao (2016) via standard fixed point method. In this paper, three more elegant iterative methods are proposed to find the approximate solution of the nonlinear matrix equation $X - A^*e^XA = I$, namely: Newton's method; modified fixed point method and a combination of Newton's method and fixed point method. The convergence of Newton's method and modified fixed point method are derived. Comparative numerical experimental results indicate that the new developed algorithms have both less computational time and good convergence properties when compared to their respective standard algorithms.

**Keywords:** Hermitian positive definite solution, nonlinear matrix equation, modified fixed point method, iterative method.

## Introduction

The nonlinear matrix equation

$$X - A^*e^XA = I \qquad (1)$$

is considered, where $A$ is the given square matrix, $A^*$ denotes the matrix of complex conjugate entries, $I$ is an identity matrix and $X$ is an unknown Hermitian Positive Definite Solution (HPDS) to be determined. The general basic form of Equation (1) is

$$X + A^*\mathcal{F}(X)A = Q, \quad \text{where} \quad Q > 0. \qquad (2)$$

Equation (2) for different $\mathcal{F}(X)$ has been studied widely (Engwerda 1993, Zhang et al. 2011, Chacha and Kim 2019) and has been found to be applicable in modeling of physical processes arising in statistics, control theory, stochastic filtering and Kalman filtering (Anderson et al. 1990, Engwerda 1993, Guo and Lancaster 1999, Ivanov et al. 2005, Berzing and Samet 2011). Chacha and Naqvi (2018) derived the condition numbers of the nonlinear matrix equation $X^n - A^*e^XA = I$ and developed an iterative algorithm useful in finding its approximate solution. Gao (2016) derived sufficient and necessary conditions for the existence of HPD solution of Equation (1) and suggested the basic fixed point method to obtain its approximate HPD solution. However, no numerical experiments were reported to reveal the efficiency of the developed algorithm.

This study is important in the following ways: First, a Newton's method is applied in finding the solution of Equation (1). Second, it introduces the modified fixed point algorithm and a combined Newton's method and the fixed point method algorithm in obtaining solution of Equation (1). It is further shown that modified fixed point algorithm outperforms Newton's method in terms of computation time especially for large matrix size. Moreover, a combination of pure Newton's method and fixed point method is proposed and outperform pure Newton's method since the Newton's step is solved by the basic fixed point approach which makes it to take less computation time. It also derives the existence of a fixed point by applying Banach's fixed point theorem and shows that the modified fixed point method has a better convergence as compared to the basic fixed point method. The following notations and definitions will be used throughout this paper: $\rho(\blacksquare)$ stands for spectral radius: $\mathcal{F} \circ \mathcal{M}$ denotes the composition of the operators $\mathcal{F}$ and $\mathcal{M}$, where $\mathcal{F} \circ \mathcal{M}(X) = \mathcal{F}(\mathcal{M}(X))$; $vec\,(A) = [a_1{}^T, a_2{}^T. \cdots, a_n{}^T]^T$ is the column-wise vector representation of matrix $A$ and $vec(AXB)$=$(B^T \otimes A)vec(X)$; $C \otimes D = [c_{ij}]B$ is the tensor or kronecker product of the matrices $C$ and $D$; $\overline{B_\epsilon(X_0)}$ stands for a closed ball with a radius $\epsilon$ and centre $X_0$; $A^T$ stands for transpose of matrix $A$; 0 represents square null matrix; $\oplus$ stands for Kronecker sum; $\|\blacksquare\| := \|\blacksquare\|_2$ is the spectral norm; $\|\blacksquare\|_F$ stands for Frobenius norm and for any matrices $C, D \in \mathbb{R}^{m\times n}$, $C \geq D (C > D)$ if $[c_{ij}] \geq [d_{ij}]([c_{ij}] > [d_{ij}])$ for all $i, j$.

**Definition 1** (Ortega and Rheinboldt 2008)
For a general function $F: C^{n\times n} \to C^{n\times n}$, Newton's method for the solution of $F(X) = 0$ is specified by an initial approximation $X_0$ and the recurrence $X_{k+1} = X_k - F'(X_k)^{-1}F(X_k)$, for all $k = 0, 1, 2, \cdots$, where $F'$ denotes the Fréchet derivative.

**Definition 2** (Higham and Al-Mohy 2008, Mathias 1992)
The Fréchet derivative of a matrix function $e^X$ at $X_0$ in the direction $Z$ is given by

$$\int_0^1 e^{tX}Ze^{(1-t)X}dt \approx e^{X/2}Ze^{X/2} \qquad (3)$$

**Lemma 1** (Theorem 3, Gao 2016)
If $A$ is invertible and equation (1) has a solution, then $\rho(A) \leq \frac{1}{e}$.

Lemma 2 (Ortega and Rheinboldt 2008)
Let $A, B \in C^{n\times n}$ and **assume** that $A$ is invertible with $\|A^{-1}\| \leq \alpha$. If $\|A - B\| \leq \beta$ and $\alpha\beta < 1$, then $B$ is also invertible, and $\|B^{-1}\| \leq \frac{\alpha}{(1-\alpha\beta)}$.

**Lemma 2**
Let $X, Y \in C^{n\times n}$, then $\|e^X - e^Y\| \leq \|X - Y\|e^{\max(\|X\|, \ \|Y\|)}$.

Proof: **From** the exponential identity $e^{(A+Z)t} = e^{Xt} + \int_0^1 e^{X(t-s)}Ze^{(X+Z)s}ds$, with $t = 1$, and $Y = X + Z$, it follows that

$$e^Y = e^X - \int_0^1 e^{X(1-s)}(Y - X)e^{Ys}ds = e^X + \int_0^1 e^{X(1-s)}(X - Y)e^{Ys}ds$$

**Thus**,

$$\|e^X - e^Y\| \leq \|X - Y\| \int_0^1 e^{\|X\|(1-s)}e^{\|Y\|s}ds \leq \|X - Y\| \int_0^1 e^{\max(\|X\|, \ \|Y\|)}ds = \|X - Y\|e^{\max(\|X\|, \ \|Y\|)}.$$

**This completes the proof of lemma**.

## Materials and Methods

In this section, Newton's method, a CNFP method (a combination of Newton's method and fixed point method) and modified fixed point method are proposed.

### Newton's method

Let Equation (1) be represented by the map

$$F(X) = X - A^*e^XA - I \qquad (4).$$

Applying **Definition 2** on Equation (4), one has that

$$F(X + Z) = X + Z - [A^*(e^{X+Z} - e^X)A] - A^*e^XA - I$$
$$= X - A^*e^XA - I + Z - [A^*(e^{X+Z} - e^X)A] + O(Z^2)$$
$$F(X + Z) = F(X) + Z - A^*e^{\frac{X}{2}}Ze^{\frac{X}{2}}A + O(Z^2) \qquad (5)$$

From Equation (5), the Fréchet derivative $F_X'$ for the function $F(X)$ is obotained.

From **Definition 1**, we know that $F'_X(Z): C^{n\times n} \to C^{n\times n}$ is the linear operator defined by $F'_X(Z) = Z - A^* e^{\frac{X}{2}} Z e^{\frac{X}{2}} A$ and $vec(F'_X(Z)) = \left(I_{n^2} - \left(e^{\frac{X}{2}}A\right)^T \otimes \left(A^* e^{\frac{X}{2}}\right)\right) vec(Z)$, we get the tensor Fréchet derivative denoted by $D_X = I_{n^2} - \left(e^{\frac{X}{2}}A\right)^T \otimes \left(A^* e^{\frac{X}{2}}\right)$. By **Lemma 1**, we know that $\rho(D_X) < 1$ and $D_X$ is non-singular. Thus, the matrix sequence generated by Newton's method for Equation (1) is given by

$$\begin{cases} Z_i - A^* e^{\frac{X_i}{2}} Z e^{\frac{X_i}{2}} A = -F(X_i), \\ X_{i+1} = X_i + Z_i. \end{cases} \quad \text{for all} \quad i = 0,1,2,\cdots \tag{6}$$

Iteration (6) can be rewritten as

$$X_{i+1} = X_i - \left(D_{X_i}\right)^{-1}\left(F(X_i)\right) \quad \text{for all } i = 0,1,2,\cdots, \text{ which is equivalent to}$$

$$X_{i+1} - A^* e^{\frac{X_i}{2}} X_{i+1} e^{\frac{X_i}{2}} A = -A^* e^{\frac{X_i}{2}} X_i e^{\frac{X_i}{2}} A + A^* e^{X_i} A + I.$$

**Newton's method (NM) for equation (1)**

**Step 1:** Given a symmetric matrix $A$ and an initial guess $X_0$.

**Step 2:** Solve Newton's step in $Z_i - A^* e^{\frac{X_i}{2}} Z e^{\frac{X_i}{2}} A = -F(X_i)$.

**Step 3:** $X_{i+1} = X_i + Z_i$, for all $i = 0, 1, 2, \cdots$.

**Step 4:** Check if $\|F(X_k)\|_F \le n.\text{eps}$, where $n$ is the size of matrix $A$ and eps= $2.22\times 10^{-16}$, otherwise go to **Step 2**.

Step 5: Display the approximate solution $X$.

**Convergence of Newton's method for Equation (1)**

In this subsection, it is shown that NM converges to the solution $X$.

**Lemma 3**

Let $X$ be invertible with $\|A\|^2 \left\|e^{\frac{X}{2}}\right\|^2 < 1$. Then, the linear operator $F_X{}'$ is non-singular, and $\|(F_X{}')^{-1}\| \le \frac{1}{\left(1-\|A\|^2\left\|e^{\frac{X}{2}}\right\|^2\right)}$.

Proof: From $F'_X(Z) = Z - A^* e^{\frac{X}{2}} Z e^{\frac{X}{2}} A$, it follows that $\|F'_X(Z)\| \ge \left(1 - \|A\|^2 \left\|e^{\frac{X}{2}}\right\|^2 \|Z\|\right)$. From the assumption that $\|A\|^2 \left\|e^{\frac{X}{2}}\right\|^2 < 1$, it follows that $F'_X(Z) = 0$ if and only if $Z = 0$.

This is to say that the operator $F'_X$ is injective. Since $F'_X$ is an operator on the finite dimension linear space, $F'_X$ is surjective. It follows that $F'_X$ is regular, and $\|(F'_X)^{-1}\| = \frac{1}{\min\left\{\frac{\|F'_X\|}{\|Z\|}: Z\neq 0\right\}} \le \frac{1}{\left(1-\|A\|^2\left\|e^{\frac{X}{2}}\right\|^2\right)}$.

**Theorem 1**

**Suppose** that $X_0 \in \mathbb{R}^{n\times n}$ is invertible, and the mapping $F$ defined in Equation (4) is local**ly** Lipschitz continuous in the neighborhood of $X_0$. More accurately, there exists $\epsilon > 0$ and $K > 0$, such that for all $Y \le X$ in $\overline{B_\epsilon(X_0)}$, it holds that $\|F'_X - F'_Y\| \le K\|X - Y\|$, where $B_\epsilon(X_0) = \{X: \|X - X_0\| \le \epsilon\}$ and $F'_X$, $F'_Y$ are the Fréchet derivatives of $F$ in Equation (4) at $X, Y \in \mathbb{R}^{n\times n}$ and $K = \frac{\|A\|^2{}_F}{2}\left\|(X - Y)^{-1}\left((X\oplus X) - (Y\oplus Y)\right)\right\|_2 \|vec(Z)\|_2 e^{\max\left\{\left\|\frac{X}{2}\oplus\frac{X}{2}\right\|_{2,}\ \left\|\frac{X}{2}\oplus\frac{X}{2}\right\|_2\right\}}$

Proof: Suppose $F'_X(Z)$ and $F'_Y(Z)$ are well defined. Employing **Lemma 2** and the properties of tensor product and Kronecker sum, it follows that

$\|F'_X(Z) - F'_Y(Z)\|_F$

$$= \left\| A^{*}e^{\frac{X}{2}}Ze^{\frac{X}{2}}A - A^{*}e^{\frac{Y}{2}}Ze^{\frac{Y}{2}}A \right\|_{F}$$
$$\leq \|A\|^{2}{}_{F} \left\| \left(e^{\frac{X}{2}} \otimes e^{\frac{X}{2}}\right) vec(Z) - \left(e^{\frac{Y}{2}} \otimes e^{\frac{Y}{2}}\right) vec(Z) \right\|_{2}$$
$$\leq \|A\|^{2}{}_{F} \left\| e^{\left(\frac{X}{2} \oplus \frac{X}{2}\right)} - e^{\left(\frac{Y}{2} \oplus \frac{Y}{2}\right)} \right\|_{2} \|vec(Z)\|_{2}$$
$$\leq \|A\|^{2}{}_{F} \left\| \left(\frac{X}{2} \oplus \frac{X}{2}\right) - \left(\frac{Y}{2} \oplus \frac{Y}{2}\right) \right\|_{2} e^{\max\left\{\left\|\frac{X}{2} \oplus \frac{X}{2}\right\|_{2,} \ \left\|\frac{Y}{2} \oplus \frac{Y}{2}\right\|_{2}\right\}} \|vec(Z)\|_{2}$$
$$= \frac{\|A\|^{2}{}_{F}}{2} \|(X \oplus X) - (Y \oplus Y)\|_{2} \|vec(Z)\|_{2} e^{\max\left\{\left\|\frac{X}{2} \oplus \frac{X}{2}\right\|_{2,} \ \left\|\frac{Y}{2} \oplus \frac{Y}{2}\right\|_{2}\right\}}$$

$$= \frac{\|A\|^{2}{}_{F}}{2} \left\|(X - Y)(X - Y)^{-1}\big((X \oplus X) - (Y \oplus Y)\big)\right\|_{2} \|vec(Z)\|_{2} e^{\max\left\{\left\|\frac{X}{2} \oplus \frac{X}{2}\right\|_{2,} \ \left\|\frac{Y}{2} \oplus \frac{Y}{2}\right\|_{2}\right\}}$$

$$= \frac{\|A\|^{2}{}_{F}}{2} \left\|(X - Y)^{-1}\big((X \oplus X) - (Y \oplus Y)\big)\right\|_{2} \|vec(Z)\|_{2} e^{\max\left\{\left\|\frac{X}{2} \oplus \frac{X}{2}\right\|_{2,} \ \left\|\frac{Y}{2} \oplus \frac{Y}{2}\right\|_{2}\right\}} \|X - Y\|_{2}.$$

Finally, we have

$\|F'_X - F'_Y\|_F \leq K\|X - Y\|_2$, where $K = \frac{\|A\|^{2}{}_{F}}{2} \left\|(X - Y)^{-1}\big((X \oplus X) - (Y \oplus Y)\big)\right\|_{2} e^{\max\left\{\left\|\frac{X}{2} \oplus \frac{X}{2}\right\|_{2,} \ \left\|\frac{Y}{2} \oplus \frac{Y}{2}\right\|_{2}\right\}}$, which ends the proof of the theorem.

**Theorem 2**

Suppose that Equation (1) has a nonsingular solution $X^{\text{sol.}}$ and the mapping $F'_{X^{\text{sol.}}}$ is invertible. Then, there exists a closed ball $P = \overline{B_{\epsilon}(X^{\text{sol.}})}$, such that for all $X_0 \in P$, the sequence $X_k$ generated by Newton's method (NM) converges quadratically to the solution $X^{\text{sol.}}$.

Proof: Let $\psi(X) = X - (F'_X)^{-1}F(X)$. Applying Taylor's formula for Banach space in Guo (2009), we have

$$\lim_{\|Z\| \to 0} \frac{\left\|\psi\left(X^{\text{sol.}} + Z\right) - \psi\left(X^{\text{sol.}}\right)\right\|}{\|Z\|}$$
$$= \lim_{\|Z\| \to 0} \left( \left\| X^{\text{sol.}} + Z - \left(F'_{X^{\text{sol.}}+Z}\right)^{-1} F(X^{\text{sol.}} + Z) - \left(X^{\text{sol.}} - \left(F'_{X^{\text{sol.}}}\right)^{-1} F(X^{\text{sol.}})\right) \right\| \times \|Z\|^{-1} \right)$$
$$= \lim_{\|Z\| \to 0} \left( \left\| Z + \left(\left(F'_{X^{\text{sol.}}}\right)^{-1} F\left(X^{\text{sol.}}\right)\right) - \left(F'_{X^{\text{sol.}}+Z}\right)^{-1} F(X^{\text{sol.}} + Z) \right\| \times \|Z\|^{-1} \right)$$

$$= \lim_{\|Z\| \to 0} \left( \left\| Z + \left(\left(F'_{X^{\text{sol.}}}\right)^{-1} F\left(X^{\text{sol.}}\right)\right) - \left(F'_{X^{\text{sol.}}+Z}\right)^{-1} \left[ F\left(X^{\text{sol.}}\right) + F'_{X^{\text{sol.}}}(Z) + \frac{1}{2} F''_{X^{\text{sol.}}}(Z^2) + \cdots \right] \right\| \times \|Z\|^{-1} \right) = 0.$$

This implies that the Fréchet derivative of $\psi$ at $X^{\text{sol.}}$ is zero. Applying Ostrowski Theorem (Ortega 1972) and Theorem 1, it can be proved that the matrix sequence $\{X_k\} \in P$ produced by NM algorithm converges quadratically to the solution $X^{\text{sol.}}$. In other words, $\{X_k\} \to X^{\text{sol.}}$ as $k \to \infty$.

Let $\beta \coloneqq \left\|\left(F'_{X^{\text{sol.}}}\right)^{-1}\right\|$. From Lemma 1, we choose $0 < \gamma < \beta^{-1}$. It follows that

$$\|(F'_X)^{-1}\| = \frac{\beta}{1 - \beta\gamma}.$$

From Theorem 1, we have

$\left\|F'_{X_k}(X_k - X^{\mathrm{sol.}}) - F'_{X^{\mathrm{sol.}}}(X_k - X^{\mathrm{sol.}})\right\| \le K\left\|X_k - X^{\mathrm{sol.}}\right\|^2$.

Similarly, employing Newton's Leibniz formula and Theorem 1, it follows that

$$\left\|F'_{X_k}(X_k - X^{\mathrm{sol.}}) - F'_{X^{\mathrm{sol.}}}(X_k - X^{\mathrm{sol.}})\right\|$$
$$= \left\|\int_0^1 \left(F'_{(1-t)X^{\mathrm{sol.}}+tX_k} - F'_{X^{\mathrm{sol.}}}\right)(X_k - X^{\mathrm{sol.}})dt\right\|$$
$$\le \left\|X_k - X^{\mathrm{sol.}}\right\| \int_0^1 \left\|F'_{(1-t)X^{\mathrm{sol.}}+tX_k} - F'_{X^{\mathrm{sol.}}}\right\| dt$$
$$\le K/2\left\|X_k - X^{\mathrm{sol.}}\right\|^2.$$

From the matrix sequence produced by NM for Equation (1), it follows that

$$\left\|X_k - X^{\mathrm{sol.}}\right\| = \left\|X_k - (F'_{X_k})^{-1}(F(X_k)) - X^{\mathrm{sol.}}\right\|$$
$$= \left\|(F'_{X_k})^{-1}F'_{X_k}(X_k - X^{\mathrm{sol.}}) - F(X_k)\right\|$$
$$= \left\|(F'_{X_k})^{-1}\left(F'_{X_k}(X_k - X^{\mathrm{sol.}}) - F'_{X^{\mathrm{sol.}}}(X_k - X^{\mathrm{sol.}}) - \left(F(X_k) - F(X^{\mathrm{sol.}}) - F'_{X^{\mathrm{sol.}}}(X_k - X^{\mathrm{sol.}})\right)\right)\right\|$$
$$\le \left\|(F'_{X_k})^{-1}\right\| \left\|F'_{X_k}(X_k - X^{\mathrm{sol.}}) - F'_{X^{\mathrm{sol.}}}(X_k - X^{\mathrm{sol.}})\right\| + \left\|\left(F(X_k) - F(X^{\mathrm{sol.}}) - F'_{X^{\mathrm{sol.}}}(X_k - X^{\mathrm{sol.}})\right)\right\|$$
$$\le \frac{\beta K}{1-\beta\gamma}\left\|X_k - X^{\mathrm{sol.}}\right\|^2 + \frac{\beta K}{2(1-\beta\gamma)}\left\|X_k - X^{\mathrm{sol.}}\right\|^2 = M\left\|X_k - X^{\mathrm{sol.}}\right\|,^2$$

where $M = \frac{3\beta K}{2(1-\beta\gamma)}$. Hence, $\lim_{k\to\infty} X_k = X^{\mathrm{sol.}}$.

This marks the end of the proof of the theorem.

Newton's method is among the powerful methods involved in solving equations of the form $\mathcal{F}(X) = 0$. Also, in some cases it may give better convergence by reducing the number of iterative steps when incorporated with line searches depending on the nature of the problem (Kim 2000, Higham and Kim 2001). Seo and Kim (2014) employed relaxed Newton's method to solve a matrix polynomial equation and obtained relatively quicker convergence as compared to that of pure Newton's method.

A new algorithm is provided which is a combination of pure Newton's method and fixed point method (CPNFP). Unlike in pure NM method in which solving Newton's step involves computation of Kronecker product, CPNFP method solves Newton's step by a fixed point method incorporating composite function.

Now, a new CPNFP method is provided for solving an approximate solution for Equation (1) as follows:

Let $Z_i := G(Z_i, X_i) = A^* e^{\frac{X_i}{2}} Z e^{\frac{X_i}{2}} A - F(X_i)$.

Step 1: Given matrices, $Z_0, X_0$.

Step 2: $Z_i := G(Z_{i-1}, X_{i-1})$; for all $i = 1, 2, \cdots$

Step 3: $Z_{i+1} := G(Z_i, X_i)$

Step 4: $X_k = X_{i-1} + Z_{i+1}$

**Step 5:** Check if $\|F(X_k)\|_{\mathrm{F}} \le n.\mathrm{eps}$ or $\|Z_{i+1} - Z_i\|_F \le n.\mathrm{eps}$, where $n$ is the size of matrix $A$ and eps= $2.22\times 10^{-16}$, otherwise go to **Step 2**.

**Step 6:** Display the approximate solution $X$.

**Fixed point method (FPM)**

Gao (2016) provided a fixed point method for Equation (1) as follows:

Given $X_0 \in S = [I, 2I]$, $X_i = I + A^* e^X A$, for all $i = 1,2,\cdots$

In this work, a composite function is incorporated in the fixed point iteration to accelerate the convergence and reduce computational time.

Let $X_i := H(X_i) = I + A^*e^{X_i}A$, then the new matrix sequence is produced by the sequence $X_{i+1} := H \circ H(X_i) = H\big(H(X_i)\big)$. We now have the modified fixed point method for equation (1).

**Modified fixed point method (MFP) for Equation (1)**

Step 1: Given matrix $A$, choose $X_0 \in S = [I, 2I]$

Step 2: $X_i := H(X_i)$; for all $i = 0,1,2,\cdots$

Step 3: $X_{i+1} := H(H(X_i))$

Step 4: Check if $\|F(X_i)\|_{\mathrm{F}} \leq n.\mathrm{eps}$ or $\|X_{i+1} - X_i\|_F \leq n.\mathrm{eps}$ , where $n$ is the size of matrix $A$ and eps= $2.22\times 10^{-16}$, otherwise go to **Step 2**.

Step 6**:** Display the approximate solution $X$.

**Theorem 3**

Let $C \in \mathbb{C}^{n\times n}$ be an ordered set such that every pair $X, Y \in C = [I, 2I]$ has lower and upper bound. Moreover, suppose that $d(X, Y) = \|X - Y\|$ is a metric on $C$ such that $(C, d)$ is a complete metric space. If $H$ is continuous monotone and there exists $\delta \in (0,1)$ such that $d(H\big(H(X)\big), H\big(H(Y)\big)) \leq \delta d(X, Y)$, where $X \geq Y$, then $H$ has a fixed point $X^{sol.}$, where $H(X) = I + A^*e^XA$ and $\delta = \|A\|^4e^{2\|X\|}$.

Proof: Let $H(X) = I + A^*e^XA$ be continuous monotone and suppose that the pair $X, Y \in C = [I, 2I]$ is well ordered. We have that,

$$\begin{aligned}
&\big\|H\big(H(X)\big),\quad H\big(H(Y)\big)\big\| \\
&= \big\|A^*e^{(I+A^*e^XA)}A - A^*e^{(I+A^*e^YA)}A\big\| \\
&\leq \|A\|^2\big\|e^{(I+A^*e^XA)} - e^{(I+A^*e^YA)}\big\| \\
&= \|A\|^2\big\|e^{H(X)} - e^{H(Y)}\big\| \\
&\leq \|A\|^2\|H(X) - H(Y)\|e^{\max\{\|H(X)\|,\quad \|H(Y)\|\}} \\
&\leq \|A\|^4\|e^X - e^Y\|e^{\max\{\|H(X)\|,\quad \|H(Y)\|\}} \\
&\leq \|A\|^4e^{2\|X\|}\|X - Y\|.
\end{aligned}$$

Thus, the desired result $d(H\big(H(X)\big), H\big(H(Y)\big)) \leq \delta d(X, Y)$ , where $\delta = \|A\|^4e^{2\|X\|}$ is achieved.

Consequently, from Banach fixed point theorems (Ran and Reurings 2003, Sawangsup and Sintunavarat 2017), $H$ has a unique fixed point.

It is easy to see that $\lim\limits_{i\to\infty} X_i = X^{sol.}$.

**Results and Discussion**

In this section, numerical tests are used to illustrate the effectiveness of the suggested algorithms in comparison with the previously suggested algorithm to solve Equation (1). In **Example 1,** a $4\times 4$ matrix $A$ is provided and employed four algorithms proposed to compute the solution of Equation (1). The summary of results is presented in Table 1 and Table 2. In **Example 2,** 13 matrices with different sizes $(n)$ are provided and employed four algorithms to find the solution of Equation (1). The summary of results is presented in Table 3. The experiments were done in MATLAB R2015a and the loops were terminated whenever the error $\|F(X)\|_F \leq n.\mathrm{eps}$, where $n$ is the matrix size and eps $= 2.22\times 10^{-16}$.

Example 1: Consider Equation (1) with matrix $A = \frac{1}{8}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ and an initial solution $X_0 = 1.2I$. Then, employ NM, CNMFP, FP and MFP algorithms to compute the solution $X$ of Equation (1).

The solution $X^{\mathrm{sol.}} = \begin{bmatrix} 1.1047 & 0 & 0 & 0.1047 \\ 0 & 0.1047 & 0.1047 & 0 \\ 0 & 0 & 1 & 1 \\ 0.1047 & 0 & 0 & 0.1047 \end{bmatrix}$ for all the four algorithms in four decimal places.

**Table 1:** Results summary for Example 1, $X_0 = 1.2I$.

| Method | Iteration | error | CPU time in seconds |
|---|---|---|---|
| NM | 4 | $1.1444 \times 10^{-16}$ | 0.0360 |
| CNMFP | 18 | $2.9894 \times 10^{-16}$ | 0.0631 |
| FP | 20 | $2.6200 \times 10^{-16}$ | 0.0816 |
| MFP | 10 | $2.6200 \times 10^{-16}$ | 0.0398 |

**Remark 1**

Based on results provided in Table 1, NM and MFP algorithms converge faster as compared to FP and CNMFP algorithms. This may not be the case when we consider large matrix size. Thus, this is valid for small matrix sizes.

**Table 2:** Results summary for Example 1, $X_0 = 2I$.

| Method | Iteration | error | CPU time in seconds |
|---|---|---|---|
| NM | 6 | $1.1388 \times 10^{-16}$ | 0.0614 |
| CNMFP | 20 | $1.2413 \times 10^{-16}$ | 0.0601 |
| FP | 23 | $5.6200 \times 10^{-16}$ | 0.0746 |
| MFP | 12 | $1.3900 \times 10^{-16}$ | 0.0422 |

**Remark 2**

Based on results provided in Table 2, it is revealed that the choice of initial guess affects the performance of all algorithms. Newton's method is mostly affected as compared to the remaining algorithms. It also implies that the solution of Equation (1) is relatively closer to $1.2I$ than $2I$.

**Example 2:** Consider real symmetric matrix $A = \frac{\mathrm{rand}(n) + (\mathrm{rand}(n))^T}{400}$ with $X_0 = 1.2I$. Then, four algorithms are employed to compute the approximate solution of Equation (1) for matrix sizes $n = 10, 15, 20, 25, 30, 40, 45, 50, 60, 70, 80, 90$ and 100.

A summary of results for **Example 2** is presented in Table 3.

**Table 3:** A summary of results for Example 2

| $n$ | NM | | CPNMFP | | FP | | MFP | |
|---|---|---|---|---|---|---|---|---|
| | IT | CPU(Sec.) | IT | CPU(Sec.) | IT | CPU(Sec.) | IT | CPU(Sec.) |
| 10 | 4 | 0.3849 | 6 | 0.0308 | 6 | 0.0297 | 3 | 0.0227 |
| 15 | 4 | 1.7455 | 7 | 0.0726 | 6 | 0.0337 | 3 | 0.0314 |
| 20 | 4 | 6.0246 | 8 | 0.1574 | 7 | 0.0570 | 4 | 0.0551 |
| 25 | 4 | 13.3337 | 8 | 0.1650 | 7 | 0.0768 | 4 | 0.0649 |
| 30 | 4 | 28.3417 | 8 | 0.3102 | 8 | 0.1906 | 4 | 0.1061 |
| 40 | 4 | 86.2657 | 9 | 0.5728 | 9 | 0.2141 | 5 | 0.2097 |
| 45 | 4 | 139.0020 | 10 | 0.6397 | 10 | 0.2784 | 5 | 0.2471 |
| 50 | 4 | 203.1535 | 10 | 0.7650 | 10 | 0.3055 | 5 | 0.3831 |
| 60 | 4 | 432.8035 | 10 | 0.9259 | 12 | 0.5095 | 6 | 0.4534 |
| 70 | 4 | 600.7411 | 12 | 2.3360 | 13 | 0.7429 | 7 | 0.7425 |
| 80 | 4 | 879.3148 | 13 | 2.1954 | 15 | 1.0697 | 7 | 0.8743 |
| 90 | 4 | 1781.3273 | 15 | 3.2106 | 16 | 1.4201 | 8 | 1.2598 |
| 100 | 4 | 3526.7338 | 16 | 4.0318 | 18 | 1.9152 | 9 | 1.7888 |

**IT stands for iterations**

**Remarks 3:** Results in Table 3 reveal that NM method is the worst when compared to the remaining methods in terms of CPU time. This is due to the computation of tensor product when calculating Newton's step at every iterative step. Modified fixed point method has the best performance in terms of CPU time.

**Conclusion**

NM, CPNMFP, FP and MFP iterative methods have been proposed for solving Equation (1). It is revealed that NM method has the best performance in terms of CPU time for small matrix sizes. On the other hand, NM method has the worst CPU performance when large matrix sizes are involved in Equation (1). CPNMFP and MFP iterative methods are very effective when dealing with large matrix sizes. MFP method has the best performance in terms of CPU time when matrix $A$ in Equation (1) is very large (size $(A) \geq 10$).

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